Vol. 50 (2002) REPORTS ON MATHEMATICAL PHYSICS No. 3

A CHARACTERIZATION OF STRICT JACOBI–NIJENHUIS MANIFOLDS THROUGH THE THEORY OF LIE ALGEBROIDS

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(*Received January 8, 2002 – Revised March 11, 2002*)

We obtain a characterization of strict Jacobi–Nijenhuis structures using the equivalent notions of generalized Lie bialgebroid and Jacobi bialgebroid.

Keywords: Jacobi–Nijenhuis manifold, Poisson–Nijenhuis manifold, Lie algebroid, Lie bialgebroid.

1. Introduction

The notion of Jacobi–Nijenhuis manifold was introduced in [12] by J. C. Marrero, J. Monterde and E. Padrón as a generalization of the weak Poisson–Nijenhuis structure presented in [13]. In this work we introduce the notion of *strict* Jacobi–Nijenhuis manifold, which seems to be the natural generalization of the definition of Poisson–Nijenhuis manifold initially given by F. Magri and C. Morosi in [11].

When a Poisson manifold (M, Λ) is equipped with a Nijenhuis tensor N, we can associate with this manifold two Lie algebroid structures defined respectively on the tangent and on the cotangent bundles of M. Using the notion of Lie bialgebroid, which was introduced by K. Mackenzie and P. Xu in [10], Y. Kosmann-Schwarzbach showed in [7] that (M, Λ, N) is a Poisson–Nijenhuis manifold if and only if these two Lie algebroids constitute a Lie bialgebroid. Our aim is to show that a similar relation can be obtained when a differentiable manifold is equipped with a Jacobi structure and a Nijenhuis operator. For this purpose, we will use the notion of generalized Lie bialgebroid, introduced by D. Iglesias and J. C. Marrero in [2]. This notion is equivalent to the one introduced by J. Grabowski and G. Marmo in [1], under the name of Jacobi bialgebroid. Generalized Lie bialgebroids are closely related to Jacobi structures. In fact, it was proved in [2] that with each Jacobi manifold one can associate, in a certain manner, a generalized Lie bialgebroid and that the base manifold of a generalized Lie bialgebroid possesses a Jacobi structure.

Similar results to those found in this paper were obtained, independently, in [3].

* Supported by CMUC-FCT and PRAXIS.
Presented during the conference "Multi-Hamiltonian Structures: Geometric and Algebraic Aspects", 9–17 August 2001, in Mathematical Conference Center in Będlewo, Poland.

2. Lie bialgebroids and Poisson–Nijenhuis manifolds

A *Lie algebroid* $(A, [,], \rho)$ over a manifold M is a vector bundle A over M together with a bundle map $\rho : A \to TM$ and a Lie algebra structure $[,]$ on the space $\Gamma(A)$ of the global cross sections such that

(i) the map $\Gamma(\rho) : \Gamma(A) \to \mathfrak{X}(M)$, induced by ρ, is a Lie algebra homomorphism;
(ii) for any $f \in C^\infty(M)$ and $X, Y \in \Gamma(A)$,

$$[X, fY] = f[X, Y] + (\Gamma(\rho)(X).f)Y.$$

The map ρ is called the *anchor map* and usually $\Gamma(\rho)$ is denoted by ρ.

It is well known [8] that with each Lie algebroid $(A, [,], \rho)$ a differential d on the graded space of sections of $\Lambda A^* = \oplus_{k\in\mathbb{Z}}\Lambda^k A^*$ is associated, where A^* is the dual vector bundle of A. More precisely, d is a derivation of degree 1 and of square 0 of the associative graded commutative algebra $(\Gamma(\Lambda A^*), \wedge)$. Also the Lie bracket on $\Gamma(A)$ can be extended to the algebra of sections of ΛA, $\Gamma(\Lambda A) = \oplus_{k\in\mathbb{Z}}\Gamma(\Lambda^k A)$. The result is a graded Lie bracket $[,]$ which is called the *Schouten bracket* of the Lie algebroid.

Suppose that the vector bundle $(A, [,], \rho)$ and its dual vector bundle $(A^*, [,]_*, \rho_*)$ are both Lie algebroids over a manifold M. Let d (resp. d_*) denote the differential of A (resp. A^*). The pair (A, A^*) is a *Lie bialgebroid* [10] if for all $X, Y \in \Gamma(A)$,

$$d_*[X, Y] = [d_*X, Y] + [X, d_*Y]. \tag{1}$$

(Equivalently, (A, A^*) is a Lie bialgebroid if d_* is a derivation of $(\Gamma(\Lambda A), [,])$, see [6]).

This notion is self-dual, in the sense that if (A, A^*) is a Lie bialgebroid so is (A^*, A), cf. [6, 10].

EXAMPLE 1. Let (M, Λ) be a Poisson manifold and $\Lambda^\sharp : T^*M \to TM$ the vector bundle morphism given by $\langle\beta, \Lambda^\sharp(\alpha)\rangle = \Lambda(\alpha, \beta)$ for all 1-forms α and β in M. Then the pair $((T^*M, [,]_\Lambda, \Lambda^\sharp), (TM, [,], \mathrm{Id}_{TM}))$ is a Lie bialgebroid over M, where $[,]_\Lambda$ is the Lie bracket of 1-forms given, for all $\alpha, \beta \in \Omega^1(M)$, by

$$[\alpha, \beta]_\Lambda = \mathcal{L}_{\Lambda^\sharp(\alpha)}\beta - \mathcal{L}_{\Lambda^\sharp(\beta)}\alpha - d(\Lambda(\alpha, \beta)). \tag{2}$$

The differential of $(TM, [,], \mathrm{Id}_{TM})$ is the de Rham differential, while the differential of $(T^*M, [,]_\Lambda, \Lambda^\sharp)$ is the Lichnerowicz–Poisson differential $d_\Lambda = [\Lambda, .]$

The previous example shows a relation between Poisson manifolds and Lie bialgebroids. Another link relating these two structures is the following [10]: if (A, A^*) is a Lie bialgebroid over M, there exists on M an induced Poisson structure,

$$\{f, h\} = \langle df, d_*h\rangle, \qquad f, h \in C^\infty(M).$$

DEFINITION 1. [11] A *Poisson–Nijenhuis manifold* (M, Λ, N) is a Poisson manifold (M, Λ) equipped with a tensor field N of type $(1, 1)$ with vanishing Nijenhuis torsion, i.e. a Nijenhuis tensor, satisfying the following compatibility conditions:

(i) $N\Lambda^\sharp = \Lambda^\sharp . {}^tN$ and
(ii) $C(\Lambda, N) = 0$, where

$$C(\Lambda, N)(\alpha, \beta) = [\alpha, \beta]_{N\Lambda} - [{}^tN\alpha, \beta]_\Lambda - [\alpha, {}^tN\beta]_\Lambda + {}^tN[\alpha, \beta]_\Lambda, \tag{3}$$

for all $\alpha, \beta \in \Omega^1(M)$, ${}^tN$ stands for the transpose of N and $[,]_\Lambda$ (resp. $[,]_{N\Lambda}$) is the bracket (2) associated with Λ (resp. $N\Lambda$).

We should remark that condition (ii) of Definition 1 can be weakened, as it was done in [13], to obtain the so-called *weak Poisson–Nijenhuis manifold.*

It is well known [8] that when N is a Nijenhuis tensor on M, the triple $(TM, [,]_N, N)$ is a Lie algebroid, where $[,]_N$ is given by

$$[X, Y]_N = [NX, Y] + [X, NY] - N[X, Y], \qquad X, Y \in \mathfrak{X}(M). \tag{4}$$

The next theorem gives a characterization of Poisson–Nijenhuis manifolds using the notion of Lie bialgebroid.

THEOREM 1. [7] *Let (M, Λ) be a Poisson manifold and N a Nijenhuis tensor on M. Then (M, Λ, N) is a Poisson–Nijenhuis manifold if and only if the pair*

$$((TM, [,]_N, N), (T^*M, [,]_\Lambda, \Lambda^\sharp))$$

is a Lie bialgebroid.

3. Jacobi bialgebroids and Jacobi manifolds

We recall that a Jacobi structure on a manifold M is a pair (Λ, E), where Λ is a bivector and E is a vector field such that $[\Lambda, \Lambda] = -2E \wedge \Lambda$ and $[E, \Lambda] = 0$.

Let (M, Λ, E) be a Jacobi manifold. Denote by $(\Lambda, E)^\# : T^*M \times \mathbb{R} \to TM \times \mathbb{R}$ the vector bundle morphism given by $(\Lambda, E)^\#(\alpha, f) = (\Lambda^\#(\alpha) + fE, -\langle \alpha, E\rangle)$, for any section α of T^*M and $f \in C^\infty(M)$. In opposition to the case of a Poisson manifold, in general one cannot define a Lie algebroid structure on the cotangent bundle of a Jacobi manifold. However, in [5] it was shown that if (M, Λ, E) is a Jacobi manifold, then $(T^*M \times \mathbb{R}, [,]_{(\Lambda, E)}, \pi \circ (\Lambda, E)^\sharp)$ is a Lie algebroid over M, where $\pi : TM \times \mathbb{R} \to TM$ is the projection over the first factor and $[,]_{(\Lambda,E)}$ is the bracket given by

$$[(\alpha, f), (\beta, h)]_{(\Lambda, E)} := (\gamma, r), \tag{5}$$

with

$$\gamma := \mathcal{L}_{\Lambda^\#(\alpha)}\beta - \mathcal{L}_{\Lambda^\#(\beta)}\alpha - d(\Lambda(\alpha, \beta)) + f\mathcal{L}_E\beta - h\mathcal{L}_E\alpha - i_E(\alpha \wedge \beta),$$

$$r := -\Lambda(\alpha, \beta) + \Lambda(\alpha, dh) - \Lambda(\beta, df) + \langle f dh - h df, E\rangle.$$

The associated differential d_* is given for all $(P, Q) \in \mathcal{V}^k(M) \oplus \mathcal{V}^{k-1}(M)$ by [4]

$$d_*(P, Q) = ([\Lambda, P] + kE \wedge P + \Lambda \wedge Q, -[\Lambda, Q] + (1 - k)E \wedge Q + [E, P]), \tag{6}$$

where $\mathcal{V}^k(M) = \Gamma(\Lambda^k(TM))$.

On the other hand, if M is a differentiable manifold, then the triple $(TM \times \mathbb{R}, [\,,\,], \pi)$ is a Lie algebroid over M, where π is the projection over the first factor and $[\,,\,]$ is given by

$$[(X, f), (Z, h)] = ([X, Z], X \cdot h - Z \cdot f), \qquad (X, f), (Z, h) \in \mathfrak{X}(M) \times C^\infty(M). \quad (7)$$

The associated differential is $\mathrm{d} = (d, -d)$, d being the de Rham differential.

When (M, Λ, E) is a Jacobi manifold, a natural question that arises is whether the pair $(T^*M \times \mathbb{R}, TM \times \mathbb{R})$ is a Lie bialgebroid. The answer is no! This situation motivated the introduction, by D. Iglesias and J. C. Marrero [2], of the generalized Lie bialgebroids. The definition of generalized Lie bialgebroid was recently recast in simpler terms by J. Grabowski and G. Marmo in [1], under the name of Jacobi bialgebroid.

Let $(A, [\,,\,], \rho)$ be a Lie algebroid over M and $\theta \in \Gamma(A^*)$ a 1-cocycle for the Lie algebroid cohomology complex with trivial coefficients (see [9]), i.e. for all $X, Z \in \Gamma(A)$,

$$\theta([X, Z]) = \rho(X).(\theta(Z)) - \rho(Z).(\theta(X)). \quad (8)$$

Using the 1-cocycle θ, we can define a new representation ρ^θ of the Lie algebra $(\Gamma(A), [\,,\,])$ on $C^\infty(M)$, by setting

$$\rho^\theta : \Gamma(A) \times C^\infty(M) \to C^\infty(M), \qquad (X, f) \mapsto \rho^\theta(X, f) = \rho(X).f + \theta(X)f. \quad (9)$$

Therefore, we obtain a new cohomology complex, whose differential cohomology operator is given by

$$d^\theta : \Gamma(\Lambda^k A^*) \to \Gamma(\Lambda^{k+1} A^*), \qquad \beta \mapsto d^\theta(\beta) = d\beta + \theta \wedge \beta. \quad (10)$$

Also, for any $X \in \Gamma(A)$, the Lie derivative operator with respect to X is given by

$$\mathcal{L}^\theta_X : \Gamma(\Lambda^k A^*) \to \Gamma(\Lambda^k A^*), \qquad \beta \mapsto \mathcal{L}^\theta_X(\beta) = \mathcal{L}_X\beta + \theta(X)\beta. \quad (11)$$

It is also possible to consider a θ-Schouten bracket on the graded algebra $\Gamma(\Lambda A)$, denoted by $[\,,\,]^\theta$, which is defined as follows:

$$\begin{aligned} &[\,,\,]^\theta : \Gamma(\Lambda^p A) \times \Gamma(\Lambda^q A) \to \Gamma(\Lambda^{p+q-1} A) \\ &\quad (P, Q) \mapsto [P, Q]^\theta = [P, Q] + (p-1) P \wedge (i_\theta Q) + (-1)^p (q-1)(i_\theta P) \wedge Q. \quad (12) \end{aligned}$$

Suppose that $(A, [\,,\,], \rho)$ is a Lie algebroid over M such that in the dual bundle A^* of A there also exists a Lie algebroid structure over M, $([\,,\,]_*, \rho_*)$. Let $\theta \in \Gamma(A^*)$ (resp. $W \in \Gamma(A)$) be a 1-cocycle in the Lie algebroid cohomology complex of $(A, [\,,\,], \rho)$ (resp. $(A^*, [\,,\,]_*, \rho_*)$).

DEFINITION 2. [2] The pair $((A, \theta), (A^*, W))$ is a *generalized Lie bialgebroid* if for all $X, Z \in \Gamma(A)$ and $P \in \Gamma(\Lambda^p A)$,

1. $d_*^W[X, Z] = [d_*^W X, Z]^\theta + [X, d_*^W Z]^\theta$;
2. $(\mathcal{L}_*^W)_\theta(P) + (\mathcal{L}^\theta)_W(P) = 0$.

DEFINITION 3. [1] The pair $((A, \theta), (A^*, W))$ is a *Jacobi bialgebroid* if for all $P \in \Gamma(\Lambda^p A)$ and $Q \in \Gamma(\Lambda A)$,

$$d_*^W[P, Q]^\theta = [d_*^W P, Q]^\theta + (-1)^{p+1}[P, d_*^W Q]^\theta.$$

The equivalence of Definitions 2 and 3 was proved in [1]. Consequently, generalized Lie bialgebroids and Jacobi bialgebroids designate exactly the same structure. When $\theta = 0$ and $W = 0$, the Jacobi bialgebroid is a Lie bialgebroid.

Let (M, Λ, E) be a Jacobi manifold and let us consider the two Lie algebroids $(T^*M \times \mathbb{R}, [,]_{(\Lambda,E)}, \pi \circ (\Lambda, E)^\sharp)$ and $(TM \times \mathbb{R}, [,], \pi)$ mentioned above. Then $\theta = (0, 1)$ (resp. $W = (-E, 0)$) is a 1-cocycle of $TM \times \mathbb{R}$ (resp. $T^*M \times \mathbb{R}$) and the pair $((TM \times \mathbb{R}, \theta), (T^*M \times \mathbb{R}, W))$ is a Jacobi bialgebroid.

Similary to the relation between Lie bialgebroids and Poisson manifolds, whenever $((A, \theta), (A^*, W))$ is a Jacobi bialgebroid over M, there exists on M an induced Jacobi structure given by [2]:

$$\{f, h\} = \langle d^\theta f, d_*^W h\rangle, \qquad f, h \in C^\infty(M). \tag{13}$$

4. Jacobi bialgebroids and strict Jacobi–Nijenhuis manifolds

Let M be a C^∞-differentiable manifold and $\mathcal{N} : \mathfrak{X}(M) \times C^\infty(M) \to \mathfrak{X}(M) \times C^\infty(M)$ a $C^\infty(M)$-linear map defined, for all $(X, f) \in \mathfrak{X}(M) \times C^\infty(M)$, by

$$\mathcal{N}(X, f) = (NX + fY, \langle\gamma, X\rangle + gf), \tag{14}$$

where N is a tensor field of type $(1, 1)$ on M, $Y \in \mathfrak{X}(M)$, $\gamma \in \Omega^1(M)$ and $g \in C^\infty(M)$. $\mathcal{N} := (N, Y, \gamma, g)$ can be considered as a vector bundle map, $\mathcal{N} : TM \times \mathbb{R} \to TM \times \mathbb{R}$. We may define the *Nijenhuis torsion* $\mathcal{T}(\mathcal{N})$ of $\mathcal{N}$ with respect to the Lie bracket (7). When $\mathcal{T}(\mathcal{N})$ vanishes identically, we say that $\mathcal{N}$ is a *Nijenhuis operator* on M.

Suppose now that M is equipped with a Jacobi structure (Λ, E) and a Nijenhuis operator $\mathcal{N}$ and consider a tensor field Λ_1 of type $(2, 0)$ and a vector field E_1 on M, defined by

$$(\Lambda_1, E_1)^\# = \mathcal{N} \circ (\Lambda, E)^\#. \tag{15}$$

DEFINITION 4. A *strict Jacobi–Nijenhuis manifold* $(M, (\Lambda, E), \mathcal{N})$ is a Jacobi manifold (M, Λ, E) with a Nijenhuis operator $\mathcal{N}$ satisfying the following compatibility conditions: (i) $\mathcal{N} \circ (\Lambda, E)^\# = (\Lambda, E)^\# \circ {}^t\mathcal{N}$ and (ii) $\mathcal{C}((\Lambda, E), \mathcal{N}) = 0$, where

$$\begin{aligned}\mathcal{C}((\Lambda, E), \mathcal{N})((\alpha, f), (\beta, h)) &= [(\alpha, f), (\beta, h)]_{(\Lambda_1, E_1)} - [{}^t\mathcal{N}(\alpha, f), (\beta, h)]_{(\Lambda, E)} \\ &\quad - [(\alpha, f), {}^t\mathcal{N}(\beta, h)]_{(\Lambda, E)} + {}^t\mathcal{N}[(\alpha, f), (\beta, h)]_{(\Lambda, E)},\end{aligned} \tag{16}$$

for all $(\alpha, f), (\beta, h) \in \Omega^1(M) \times C^\infty(M)$, ${}^t\mathcal{N}$ is the transpose of $\mathcal{N}$ and $[,]_{(\Lambda,E)}$ (resp. $[,]_{(\Lambda_1,E_1)}$) is the bracket (5) associated with (Λ, E) (resp. (Λ_1, E_1)).

For more details on (strict) Jacobi–Nijenhuis manifolds, see [12] and [14].

There exists a close relation between Poisson–Nijenhuis and strict Jacobi–Nijenhuis manifolds, as the next Proposition illustrates.

PROPOSITION 2. [14] *With each strict Jacobi–Nijenhuis manifold* $(M, (\Lambda, E), \mathcal{N})$, $\mathcal{N} := (N, Y, \gamma, g)$, *a Poisson–Nijenhuis manifold* $(\tilde{M}, \tilde{\Lambda}, \tilde{N})$ *can be associated, where* $(\tilde{M}, \tilde{\Lambda})$ *is the Poissonization of* (M, Λ, E), *i.e.* $\tilde{M} = M \times \mathbb{R}$ *and* $\tilde{\Lambda} = e^{-t}(\Lambda + \frac{\partial}{\partial t} \wedge E)$, *and* $\tilde{N}$ *is the Nijenhuis tensor field on* $\tilde{M}$, *given by* $\tilde{N} = N + Y \otimes dt + \frac{\partial}{\partial t} \otimes \gamma + g\frac{\partial}{\partial t} \otimes dt$, *and conversely (t is the usual coordinate on* $\mathbb{R}$*).*

Let us now consider a differentiable manifold equipped with a Nijenhuis operator $\mathcal{N} := (N, Y, \gamma, g)$, given by (14). Using the operator $\mathcal{N}$, we may define a new bracket on $\mathfrak{X}(M) \times C^\infty(M)$, which is a deformation of the bracket (7), by setting, for all $(X, f), (Z, h) \in \mathfrak{X}(M) \times C^\infty(M)$,

$$[(X, f), (Z, h)]_{\mathcal{N}} = [\mathcal{N}(X, f), (Z, h)] + [(X, f), \mathcal{N}(Z, h)] - \mathcal{N}[(X, f), (Z, h)]. \quad (17)$$

Since the Nijenhuis torsion $\mathcal{T}(\mathcal{N})$ of $\mathcal{N}$ vanishes, the bracket $[,]_{\mathcal{N}}$ is a Lie bracket on $\mathfrak{X}(M) \times C^\infty(M)$ and $(TM \times \mathbb{R}, [,]_{\mathcal{N}}, \pi \circ \mathcal{N})$ is a Lie algebroid over M, where $\pi : TM \times \mathbb{R} \to TM$ is the projection over the first factor.

The differential of the Lie algebroid $(TM \times \mathbb{R}, [,]_{\mathcal{N}}, \pi \circ \mathcal{N})$ is $d_{\mathcal{N}} = [i_{\mathcal{N}}, \mathrm{d}]$, where $[,]$ is the graded commutator, $\mathrm{d} = (d, -d)$ with d the de Rham differential and $i_{\mathcal{N}}$ is the derivation of degree zero defined, for all $(\beta, \alpha) \in \Omega^k(M) \oplus \Omega^{k-1}(M)$, by

$$\begin{aligned} & i_{\mathcal{N}}(\beta, \alpha)((X_1, f_1), \cdots, (X_k, f_k)) \qquad (18) \\ & = \sum_{i=1}^{k} (\beta, \alpha)((X_1, f_1), \cdots, \mathcal{N}(X_i, f_i), \cdots, (X_k, f_k)), \\ & (X_1, f_1), \cdots, (X_k, f_k) \in \mathfrak{X}(M) \times C^\infty(M). \end{aligned}$$

PROPOSITION 3. *The pair* $(\gamma, g) \in \Omega^1(M) \times C^\infty(M)$ *is a* 1*-cocycle of the Lie algebroid* $(TM \times \mathbb{R}, [,]_{\mathcal{N}}, \pi \circ \mathcal{N})$.

Proof: Let (X, f) and (Z, h) be any sections of $\mathfrak{X}(M) \times C^\infty(M)$. A straightforward computation, using the fact that the Nijenhuis torsion of $\mathcal{N}$ is zero, leads to

$$\begin{aligned} (\gamma, g)([(X, f), (Z, h)]_{\mathcal{N}}) &= (NX + fY).(\langle \gamma, Z \rangle + gh) - (NZ + hY).(\langle \gamma, X \rangle + fg) \\ &= (\pi \circ \mathcal{N})(X, f).(\langle (\gamma, g), (Z, h) \rangle) \\ &\quad -(\pi \circ \mathcal{N})(Z, h).(\langle (\gamma, g), (X, f) \rangle). \end{aligned}$$

Note that $(\gamma, g) = {}^t\mathcal{N}(0, 1)$. □

Before giving our main theorem, we need to review some results from [2]. Given a Lie algebroid $(A, [,], \rho)$ over M, let us consider the vector bundle $\tilde{A} = A \times \mathbb{R} \to$

$M \times \mathbb{R}$ over $M \times \mathbb{R}$. The sections of $\tilde{A}$ can be identified with the t-dependent sections of A, t being the canonical coordinate on $\mathbb{R}$, i.e. for any $\tilde{X} \in \Gamma(\tilde{A})$ and $(x,t) \in M \times \mathbb{R}$, $\tilde{X}(x,t) = \tilde{X}_t(x)$, where $\tilde{X}_t \in \Gamma(A)$. This identification induces, in a natural way, a Lie bracket on $\Gamma(\tilde{A})$, also denoted by $[\,,\,]$:

$$[\tilde{X}, \tilde{Z}](x,t) = [\tilde{X}_t, \tilde{Z}_t](x), \qquad \tilde{X}, \tilde{Z} \in \Gamma(\tilde{A}),\ (x,t) \in M \times \mathbb{R},$$

and a bundle map, also denoted by ρ, $\rho : \tilde{A} \to T(M \times \mathbb{R}) \equiv TM \oplus T\mathbb{R}$, in such a way that $(\tilde{A}, [\,,\,], \rho)$ becomes a Lie algebroid over $M \times \mathbb{R}$.

Now, take a 1-cocycle $\theta \in \Gamma(A^*)$ and consider the following new brackets on $\Gamma(\tilde{A})$:

$$[\tilde{X}, \tilde{Z}]^{\star\theta} = \exp(-t)\left([\tilde{X}, \tilde{Z}] + \theta(\tilde{X})\left(\frac{\partial \tilde{Z}}{\partial t} - \tilde{Z}\right) - \theta(\tilde{Z})\left(\frac{\partial \tilde{X}}{\partial t} - \tilde{X}\right)\right) \tag{19}$$

and

$$[\tilde{X}, \tilde{Z}]^{-\theta} = [\tilde{X}, \tilde{Z}] + \theta(\tilde{X})\frac{\partial \tilde{Z}}{\partial t} - \theta(\tilde{Z})\frac{\partial \tilde{X}}{\partial t}, \tag{20}$$

$\tilde{X}, \tilde{Z} \in \Gamma(\tilde{A})$. Also consider the maps $\rho^{\star\theta}$, $\rho^{-\theta} : \Gamma(\tilde{A}) \to \mathcal{V}^1(M \times \mathbb{R})$ given, for any $\tilde{X} \in \Gamma(\tilde{A})$, respectively by

$$\rho^{\star\theta}(\tilde{X}) = \exp(-t)\left(\rho(\tilde{X}) + \theta(\tilde{X})\frac{\partial}{\partial t}\right) \tag{21}$$

and

$$\rho^{-\theta}(\tilde{X}) = \rho(\tilde{X}) + \theta(\tilde{X})\frac{\partial}{\partial t}. \tag{22}$$

LEMMA 4. [2] *Let $A \to M$ be a vector bundle over M, $[\,,\,] : \Gamma(A) \times \Gamma(A) \to \Gamma(A)$ a bracket on $\Gamma(A)$, $\rho : \Gamma(A) \to \mathfrak{X}(M)$ a homomorphism of $C^\infty(M)$-modules and θ a section of the dual bundle A^*. Then the following conditions are equivalent:*

(i) *$(A, [\,,\,], \rho)$ is a Lie algebroid over M and θ is a 1-cocycle,*
(ii) *$(\tilde{A}, [\,,\,]^{\star\theta}, \rho^{\star\theta})$ is a Lie algebroid over $M \times \mathbb{R}$,*
(iii) *$(\tilde{A}, [\,,\,]^{-\theta}, \rho^{-\theta})$ is a Lie algebroid over $M \times \mathbb{R}$.*

LEMMA 5. [2] *If $((A \times \mathbb{R}, [\,,\,]^{-\theta}, \rho^{-\theta}), (A^* \times \mathbb{R}, [\,,\,]_*^{\star W}, \rho_*^{\star W}))$ is a Lie bialgebroid (over $\tilde{M} = M \times \mathbb{R}$), then $((A, \theta), (A^*, W))$ is a Jacobi bialgebroid (over M), and conversely.*

THEOREM 6. *Let (M, Λ, E) be a Jacobi manifold and $\mathcal{N} =: (N, Y, \gamma, g)$ a Nijenhuis operator on M. Then $(M, (\Lambda, E), \mathcal{N})$ is a strict Jacobi–Nijenhuis manifold if and only if the pair*

$$\big(((TM \times \mathbb{R}, [\,,\,]_{\mathcal{N}}, \pi \circ \mathcal{N}), (\gamma, g)),\ ((T^*M \times \mathbb{R}, [\,,\,]_{(\Lambda,E)}, \pi \circ (\Lambda, E)^\sharp), (-E, 0))\big) \tag{23}$$

is a Jacobi bialgebroid.

Proof: From Proposition 2, $(M, (\Lambda, E), \mathcal{N})$ is a strict Jacobi–Nijenhuis manifold if and only if $(\tilde{M}, \tilde{\Lambda}, \tilde{N})$ is a Poisson–Nijenhuis manifold, which is equivalent to the fact that the pair $((T\tilde{M}, [,]_{\tilde{N}}, \tilde{N}), (T^*\tilde{M}, [,]_{\tilde{\Lambda}}, \tilde{\Lambda}^\sharp))$ is a Lie bialgebroid over $\tilde{M} = M \times \mathbb{R}$ (cf. Theorem 1).

Now, using Lemma 4 and taking into account that the map

$$\psi : (T\tilde{M}, [,]_{\tilde{N}}, \tilde{N}) \to ((TM \times \mathbb{R}) \times \mathbb{R}, [,]_{\mathcal{N}}^{-(\gamma,g)}, (\pi \circ \mathcal{N})^{-(\gamma,g)}),$$

$\psi(\tilde{X} + \tilde{f}\frac{\partial}{\partial t}) = (\tilde{X}, \tilde{f})$, and its adjoint,

$$\psi^* : ((T^*M \times \mathbb{R}) \times \mathbb{R}, [,]_{(\Lambda,E)}^{*(-E,0)}, (\pi \circ (\Lambda, E)^\sharp)^{*(-E,0)}) \to (T^*\tilde{M}, [,]_{\tilde{\Lambda}}, \tilde{\Lambda}^\sharp),$$

$\psi^*(\tilde{\alpha}, \tilde{f}) = \tilde{\alpha} + \tilde{f}dt$, are Lie algebroid isomorphisms, we may conclude that

$$(((TM\times\mathbb{R})\times\mathbb{R}, [,]_{\mathcal{N}}^{-(\gamma,g)}, (\pi\circ\mathcal{N})^{-(\gamma,g)}), ((T^*M\times\mathbb{R})\times\mathbb{R}, [,]_{(\Lambda,E)}^{*(-E,0)}, (\pi\circ(\Lambda, E)^\sharp)^{*(-E,0)}))$$

is a Lie bialgebroid over $\tilde{M} = M \times \mathbb{R}$ if and only if $(M, (\Lambda, E), \mathcal{N})$ is a strict Jacobi–Nijenhuis manifold. Finally, from Lemma 5, we get the desired result. □

PROPOSITION 7. *The Jacobi structure induced on M by the Jacobi bialgebroid $((TM\times\mathbb{R}, (\gamma, g)), ((T^*M\times\mathbb{R}, (-E, 0)))$ coincides with the one defined by $(\Lambda_1, E_1)^\# = \mathcal{N} \circ (\Lambda, E)^\#$.*

Proof: Taking into account (6) and (13), and also the equality $\langle\gamma, E\rangle = 0$ [14] we have, for all $f, h \in C^\infty(M)$,

$$\begin{aligned}\{f, h\} &= \langle d_{\mathcal{N}}^{(\gamma,g)} f, d_*^{(-E,0)} h\rangle \\ &= \langle df, (-N\Lambda^\sharp + Y \otimes E)dh\rangle - h\langle df, NE\rangle + f\langle dh, \Lambda^\sharp(\gamma) + gE\rangle.\end{aligned}$$

Since $N\Lambda^\sharp - Y \otimes E = \Lambda_1^\sharp$ and $\Lambda^\sharp(\gamma) + gE = NE = E_1$ (see [14]), the proof is complete. □

A natural question that arises is the following: can we also establish, for the weak Poisson–Nijenhuis manifolds and for the Jacobi–Nijenhuis manifolds, a similar characterization, using the Lie algebroids theory? We postpone the answer for a subsequent paper.

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